Elliptic Curves over Finite Fields

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What is an Elliptic Curve?

Definition

Let k (char $k \neq 2, 3$) be a field. An **elliptic curve** E over k is a curve defined by a polynomial of the form $y^2 = x^3 + ax + b$ with coefficients $a, b \in k$, appended with a "point at infinity." Formally, an **elliptic curve** E over k is a nonsingular, projective algebraic curve of genus 1 with points lying in $\mathbb{P}^2_k$.

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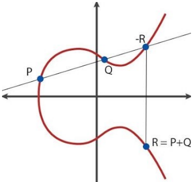
Remark

The points on an elliptic curve form a group.



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Theorem 1

$E \cong \mathbb{Z}/M\mathbb{Z} \oplus \mathbb{Z}/L\mathbb{Z}$ for unique $L, M \in \mathbb{Z}$ where $L \mid M$.

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The **order** of E is the number of elements in the group; this is $N = LM$ in the above notation. The integer M is the **group exponent**, which is the largest order of an element in the group.

Remark

Since $\mathbb{F}_q$ is a finite field, the set $E(\mathbb{F}_q)$ can be determined by iterating through all elements of $\mathbb{F}_q$ and seeing which ones satisfy the defining polynomial. (We must also remember to include the point at infinity.) This allows us to compute the group order. In practice, this might not be realistic.

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Theorem 3

Let $\#E(\mathbb{F}_q) = q + 1 - a$. Write $x^2 - ax + q = (x - \alpha)(x - \beta)$. Then

$$\#E(\mathbb{F}_{q^n}) = q^n + 1 - (\alpha^n + \beta^n)$$

for all $n \geq 1$.

Group Structure

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Let E be an elliptic curve over k. Then $E[n] \subset E(\overline{k})$ is the kernel of the map that takes point P to $P + P + \cdots + P$ (n times).

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Theorem 5

If char $k = p > 0$ and $p \mid n$, write $n = p^r n'$ with $p \nmid n'$. Then

$$E[n] \cong \mathbb{Z}/n'\mathbb{Z} \oplus \mathbb{Z}/n'\mathbb{Z} \quad \text{or} \quad \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n'\mathbb{Z}.$$

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Theorem 6

Suppose

$$E(\mathbb{F}_q) \cong \mathbb{Z}_n \oplus \mathbb{Z}_n.$$

Then either $q = n^2 + 1$ or $q = n^2 \pm n + 1$ or $q = (n \pm 1)^2$.

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Therefore, $a = q + 1 - n^2 \equiv 2 \pmod{n}$. $\square$

Proof of Theorem 6 (cont.)

Write $a = 2 + kn$. Then substituting this into $n^2 = q + 1 - a$, we have

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The possibilities $k = 0, \pm 1, \pm 2$ precisely give us the values of q in our claim.

How Balanced Can $E(\mathbb{F}_q)$ be?

Question

So we have shown that the case of $E(\mathbb{F}_q) \cong \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ is very rare. When $E(\mathbb{F}_q)$ is an unbalanced direct sum, what can we say?

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Suppose $E(\mathbb{F}_q) \cong \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/mn\mathbb{Z}$. Then $q = mn^2 + kn + 1$ for some integer k.

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Remark

This shows that in general, $E(\mathbb{F}_q)$ is substantially unbalanced. In particular, $E(\mathbb{F}_q)$ is "almost cyclic."

References

[1] Silverman, J. H. (2009). *The Arithmetic of Elliptic Curves*. New York, NY: Springer New York.

[2] Washington, L. C. (2008). *Elliptic Curves: Number Theory and Cryptography*. Boca Raton, FL: Chapman & Hall/CRC.